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LIE GROUP ANALYSIS AND CONSERVED VECTORS OF A GENERALIZED (3+1)-DIMENSIONAL KADOMTSEV-PETVIASHVILI BENJAMIN-BONA-MAHONY EQUATION WITH POWER LAW NONLINEARITY

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Abstract. In this paper, our aim is to compute exact solutions for the generalized (3+1)-dimensional Kadomtsev-Petviashvili Benjamin-Bona–Mahony (gnKP-BBM) equation by invoking an effective method, namely the Lie symmetry technique. Firstly, we derive the infinitesimals and write down the Lie symmetries. Using these symmetries the gnKP-BBM equation is reduced to various nonlinear ordinary differential equations (NLODEs). Thereafter, solutions of the NLODEs are derived by using the Jacobi elliptic cosine method, the (G'/G)-expansion method, the simplest equation technique and Kudryashov's method. Conclusively, we derive conservation laws of the gnKP-BBM equation by using the Ibragimov's method.

Keywords: Generalized (3+1)-dimensional Kadomtsev-Petviashvili Benjamin-Bona-Mahony equation; Lie point symmetries; exact solutions; conservation laws.

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1 Introduction

Nonlinear partial differential equations (NLPDEs) describe many physical phenomena of the natural world. They can exhibit complex and often unpredictable behaviour and have wide-range of applications across various fields of science and engineering (Adeyemo et al., 2023; Srivastava et al., 2021; Zhang et al., 2023; Motsepa & Khalique, (2020); Benzian, 2023; Gu, 1990; Hirota, 2004; Kudryashov, 2012; Hyder & Barakat, 2020; Wen, 2020; Zhang & Khalique, 2018; Bayrakci et al., 2023). Some common areas where they are encountered include, physics, engineering, finance and biology. In physics, NLPDEs play a pivotal role in describing the fundamental laws of physics. For example, the Navier-Stokes equations represent the motion of fluids and are essential for studying fluid dynamics, turbulence, and weather patterns. Similarly, the Schrödinger equation in quantum mechanics is a NLPDE that describes the evolution of wave

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functions in time. In biology, these equations help in many biological processes and analyse phenomena like pattern formation, cell growth, and the spread of diseases.

NLPDEs find extensive use in various engineering disciplines. They are employed to model phenomena like heat transfer, structural mechanics, fluid flow in porous media, and electromagnetic wave propagation. Solving these equations helps in designing efficient systems and optimizing performance. In finance, the Black-Scholes equation, a well-known NLPDE, is used in option pricing theory to determine the fair value of financial derivatives.

Solving NLPDEs is a challenging task due to their complex nature. Analytical solutions are often difficult or impossible to obtain, and numerical methods, such as spectral, finite element, or finite difference methods, are employed to find approximate solutions. Overall, NLPDEs provide an effective mathematical framework for understanding and analysing a wide range of natural and engineered systems, enabling scientists and engineers to make predictions and develop innovative solutions to real world problems. Therefore, various methods have been developed, such as, Bäcklund transformation (Gu, 1990), Hirota's bilinear technique (Hirota, 2004), Kudrayshov's method (Kudryashov, 2012), Darboux transformation method (Hyder & Barakat, 2020), bifurcation technique (Wen, 2020; Zhang & Khalique, 2018), sine-Gordon equation expansion approach (Chen & Yan, 2005), F-expansion approach (Zhou et al., 2018), simplest equation method (Kudryashov & Loguinova, 2018), tanh-coth approach (Wazwaz, 2018), ansatz technique (Salas & Gomez, 2018)) and Lie symmetry technique (Olver, 1993; Ovsiannikov, 1982).

Lie symmetry method is a powerful, effective and reliable mathematical tool for finding exact solutions for NLPDEs. Sophus Lie established this theory during the nineteenth century to find solutions for NLPDEs. Conservation laws also plays a crucial role in NLPDEs as they are fundamental principles that govern the behavior of physical systems. These laws express the concepts of conservation of energy, momentum, and mass. In the context of partial differential equations (PDEs), conservation laws are typically expressed as mathematical equations that express how the quantities involved in a physical system change over space and time. These equations are acquired from fundamental physical principles, such as the laws of physics or principles of conservation.

The Kadomtsev-Petviashvili (KP) equation (Kadomtsev & Petviashvili, 1970)

$$(u_t + auu_x + u_{xxx})_x + \lambda u_{yy} = 0 \tag{1}$$

was discovered in 1970 by two soviet physicists Boris Borisovich Kadomtsev and Vladimir Iosifovich Petviashvili, and was a generalization of the Korteweg and de Vries (KdV) equation. In Wazwaz (1982) the authors examined the 2D Kadomtsev-Petviashvili Benjamin-Bona-Mahony (KP-BBM) equation, namely

$$(u_x + u_t - a(u^2)_x - bu_{txx})_x + ru_{yy} = 0 \tag{2}$$

and furthermore put forward two different variants of BBM equation that were constructed in the KP sense. These two equations are

$$(u_x + u_t - a(u^n)_x - bu_{txx})_x + ru_{yy} = 0, \tag{3}$$

and

$$(u_x + u_t - a(u^{-n})_x - bu_{txx})_x + ru_{yy} = 0, \tag{4}$$

for $n > 1$. Numerous travelling wave solutions that included periodic solutions and solitons were derived for the above equations by employing the tanh and sine-cosine methods (Wazwaz, 1982). Using the expanded mapping method, Abdou (2008) was able to derive various periodic solutions, triangular wave solutions, and a solitary wave solution. Song et al. (2010) used the dynamical system bifurcation approach to discover solitary wave solutions. Using the Hirota bilinear technique, Manafianet et al. (2020) were able to construct new solutions of (2) that included lump-type solutions.

Yin et al. (2018) studied the KP-BBM equation

$$u_{tx} + \mu_1 u_{xx} + \mu_2 (uu_x)_x - \mu_3 u_{txxx} + \mu_4 u_{yy} + \mu_5 u_{zz} = 0, \tag{5}$$

where $\mu_1, \ldots, \mu_5$ are nonzero constants. The KP-BBM equation arises in various fields, including fluid dynamics, combustion, and nonlinear optics. It is a prominent example of a nonlinear wave equation and exhibits rich dynamical behavior, such as the formation and propagation of solitons. The lump-wave and breather-wave solutions were discovered through the application of bilinear forms and Hirota method.

The authors of Hoque et al. (2020) found rogue wave solutions of the KP-BBM equation

$$u_{tx} + u_{tx} + \alpha u_{xx}^2 + \beta u_{xxxt} + \sigma u_{yy} = 0$$

by using a variable transformation and the Hirota bilinear technique to the model. They also employed test functions expressed in a form of a cross-product polynomial expression. Three types of rogue wave solutions with a manageable model center and three types of rogue wave solutions with center at origin were effectively obtained. The auxiliary equation approach was employed to derive the analytical solution of equation (5) in Tariq & Seadawy (2019). In Liu (2020), the author used a symbolic computation approach to derive the first, second and third-order rogue wave solutions of the KP-BBM equation (5).

In this work, we consider power law nonlinearity of the KP-BBM equation (5), which we call generalized (3+1)-dimensional KP-BBM (gnKP-BBM) equation

$$u_{tx} + au_{xx} + b(u^n u_x)_x + cu_{txxx} + du_{yy} + eu_{zz} = 0, \tag{6}$$

where a, b, c, d, e, n are nonzero constants with $n > 0$. We perform Lie symmetry analysis of (6) and find various exact solutions of (6). Moreover, we invoke Ibragimov's method to derive its conservation laws. The paper is set out as outlined below: In Section 2 exact solutions of equation (6) are derived by using the Kudryashov's technique, Jacobi elliptic cosine technique, and (G'/G)-expansion technique. In Section 3 conservation laws of (6) are computed by invoking the Ibragimov's method. In Section 4 results and discussions are provided. Conclusively, in Section 5 we provide concluding remarks.

2 Lie symmetries of (6)

First, we find symmetries of the gnKP-BBM equation (6). The vector field

$$H = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}$$

with $\xi^i, i = 1, \ldots, 4$, η being dependent on (t, x, y, z, u) is a symmetry of (6) if

$$H^{[4]}[u_{tx} + au_{xx} + b(u^n u_x)_x + cu_{txxx} + du_{yy} + eu_{zz}] = 0, \tag{7}$$

whenever $u_{tx} + au_{xx} + b(u^n u_x)_x + cu_{txxx} + du_{yy} + eu_{zz} = 0$. The fourth extension $H^{[4]}$ of H is defined as

$$\begin{aligned} H^{[4]} = {} & X + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_z \frac{\partial}{\partial u_z} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}} \\ & + \zeta_{zz} \frac{\partial}{\partial u_{zz}} + \zeta_{txxx} \frac{\partial}{\partial u_{txxx}}, \end{aligned} \tag{8}$$

where $\zeta's$ are given by

$$\zeta_{j_1, j_2, \ldots, j_p} = D_{j_p}\left(\zeta_{j_1, j_2, \ldots, j_{p-1}}\right) - u_{j_1, j_2, \ldots, j_{p-1} k} D_{j_p}\left(\xi^k\right), \ (\text{sum on } k) \tag{9}$$

and the total differential operator is

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots . \tag{10}$$

The following eighteen linear PDEs are obtained by expanding (7) and splitting on the derivatives of u.

$$\begin{aligned}
&\xi^1_t = 0,\ \xi^1_x = 0,\ \xi^1_y = 0,\ \xi^1_z = 0,\ \xi^1_u = 0,\ \xi^2_t = 0,\ \xi^2_x = 0,\\
&\xi^2_y = 0,\ \xi^2_z = 0,\ \xi^2_u = 0,\ \xi^3_x = 0,\ \xi^3_y = 0,\ \xi^3_{zz} = 0,\ \xi^4_t = 0,\\
&\xi^4_y = 0,\ d\ \xi^4_y + e\ \xi^4_z = 0,\ \xi^4_z = 0,\ \eta = 0.
\end{aligned}$$

Solving the above PDEs, we acquire

$$\xi^1 = C_4,\ \xi^2 = C_5,\ \xi^3 = C_1 z + C_2,\ \xi^4 = -\frac{C_1 e}{d} y + C_3,\ \eta = 0$$

with $C_i, i = 1, ..., 5$ considered to be arbitrary constants. Thus, the gnKP-BBM equation (6) have the five Lie point symmetries listed below:

$$H_1 = \frac{\partial}{\partial t},\ H_2 = \frac{\partial}{\partial x},\ H_3 = \frac{\partial}{\partial y},\ H_4 = \frac{\partial}{\partial z},\ H_5 = dz\frac{\partial}{\partial y} - ey\frac{\partial}{\partial z}. \tag{11}$$

2.1 Symmetry reductions and exact solutions

2.1.1 Case 1 : Travelling wave solution using H_1, H_2, H_3 and H_4

First, we engage the symmetry $H = H_1 + H_2 + H_3 + \rho H_4$ with a constant ρ to transform the gnKP-BBM equation (6) to a PDE with three independent variables. Four invariants are obtained as a result of solving the related Lagrange system of symmetry H.

$$f = t - y,\ g = t - x,\ h = z - \rho t,\ \theta = u. \tag{12}$$

Using the above invariants, equation (6) transforms into

$$a\theta_{gg} - \theta_{fg} - \theta_{gg} + \rho\theta_{gh} + nb\theta^{n-1}\theta_g^2 + b\theta^n\theta_{gg} + c\left(\rho\theta_{gggh} - \theta_{gggg} - \theta_{fggg}\right) + d\theta_{ff} + e\theta_{hh} = 0. \tag{13}$$

The symmetries of equation (13) are

$$Y_1 = \frac{\partial}{\partial f},\ Y_2 = \frac{\partial}{\partial g},\ Y_3 = \frac{\partial}{\partial h}.$$

Solving the associated Lagrange system of $Y = Y_1 + Y_2 + \kappa Y_3$ with a constant κ, we get invariants

$$r = f - h,\ s = g - \kappa f,\ \varphi = \theta, \tag{14}$$

which transforms equation (13) into

$$\begin{aligned}
&\varphi_{ss}\left(\kappa - 1 + a - d\kappa^2\right) - \left(1 + \rho + 2d\kappa\right)\varphi_{rs} + bn\varphi^{n-1}\varphi_s^2 + b\varphi^n\varphi_{ss} - \left(c + p\right)\varphi_{rsss}\\
&\quad + \left(\kappa - 1\right)\varphi_{ssss} + \left(d + e\right)\varphi_{rr} = 0.
\end{aligned} \tag{15}$$

The NLPDE (15) has two translation symmetries, viz.,

$$Q_1 = \frac{\partial}{\partial r},\ Q_2 = \frac{\partial}{\partial s} \tag{16}$$

and as before, using $Q = Q_1 + \omega Q_2$, with a constant ω, we achieve invariants

$$\xi = r - \omega s,\ \varphi = \psi \tag{17}$$

that transform NLPDE (15) into the fourth-order nonlinear ordinary differential equation (NLODE)

$$(\omega+\kappa\omega^2-\omega^2+\rho\omega+2d\omega\kappa+d\omega^2\kappa^2+d+e)\psi''+(bn\omega^2)\psi^{n-1}\psi'^2+(b\omega^2)\psi^n\psi'' + c\omega^3(1+\kappa\omega+\rho-\omega)\psi''''=0. \tag{18}$$

Now letting

$$\psi=F^{\frac{1}{n}}(\xi),\ \xi=r-\omega s \tag{19}$$

and substituting ψ in (18) we get the NLODE

$$\left(n^2\alpha-n^3\alpha\right)F^2(\xi)F'(\xi)^2+n^3\alpha F(\xi)^3F''(\xi)+\left(n^2\beta+n^2\lambda-n^3\lambda\right)F^3(\xi)F''(\xi)^2 + n^3\lambda F^4(\xi)F''(\xi)+F'(\xi)^4\left(\mu-6n\mu+11n^2\mu-6n^3\mu\right)+\left(3n^2\mu-3n^3\mu\right)F(\xi)^2F''(\xi)^2 + \left(6n\mu-18n^2\mu+12n^3\mu\right)F(\xi)F'(\xi)^2F''''(\xi)+\left(4n^2\mu-4n^3\mu\right)F(\xi)^2F'(\xi)F''''(\xi) + n^3\mu F(\xi)^3F''''(\xi)=0. \tag{20}$$

Solution of (6) by means of Kudryashov's method

We invoke Kudryashov's method (Kudryashov, 2012) to compute exact solution of the gnKP-BBM equation (6). We presume that the solution of (20) is

$$F(\xi)=\sum_{i=0}^{M}A_iQ^i(\xi) \tag{21}$$

with $A_1,\ldots,A_M$ constants to be found, $M>0$, and $Q(\xi)$ solves the Riccati equation

$$Q'(\xi)=Q^2(\xi)-Q(\xi). \tag{22}$$

It is widely known that the solution of (22) is

$$Q(\xi)=\frac{1}{1+e^{\xi}}. \tag{23}$$

Applying the balancing method on equation (20), allows us to find M. Consequently, we get $M=2$ and the solution (21) is expressed as

$$F(\xi)=A_0+A_1Q(\xi)+A_2Q^2(\xi). \tag{24}$$

Inserting this value of $F(\xi)$ into (20), invoking (22), thereafter equating the coefficients of the like powers of Q to zero, gives an algebraic system of equations in A_0,A_1,A_2. Using Maple to solve these algebraic equations, one potential set of values for A_0,A_1,A_2 is

$$A_0=0,\ A_1=\frac{2c\left(n^2+3n+2\right)(\kappa\omega-\omega+\rho+1)}{bn^2},$$

$$A_2=-\frac{2c\left(n^2+3n+2\right)(\kappa\omega-\omega+\rho+1)}{bn^2}.$$

Thus, the solution corresponding to the above values can be written as

$$\psi(\xi)=\left\{A_1\left(\frac{1}{1+e^{\xi})}\right)+A_2\left(\frac{1}{1+e^{\xi}}\right)^2\right\}^{\frac{1}{n}}, \tag{25}$$

where $\xi = (1 + \rho + \omega\kappa - \omega)\,t + \omega x + (\omega\kappa - 1)y - z$. Subsequently, the solution of gnKP-BBM equation (6) is

$$u(t,x,y,z) = \left\{ \frac{2c\left(n^2+3n+2\right)(\kappa\omega-\omega+\rho+1)}{bn^2\left\{1+\exp((1+\rho+\omega\kappa-\omega)\,t+\omega x+(\omega\kappa-1)y-z)\right\}} - \frac{2c\left(n^2+3n+2\right)(\kappa\omega-\omega+\rho+1)}{bn^2\left\{1+\exp\left((1+\rho+\omega\kappa-\omega)\,t+\omega x+(\omega\kappa-1)y-z\right)\right\}^2} \right\}^{\frac{1}{n}}. \tag{26}$$

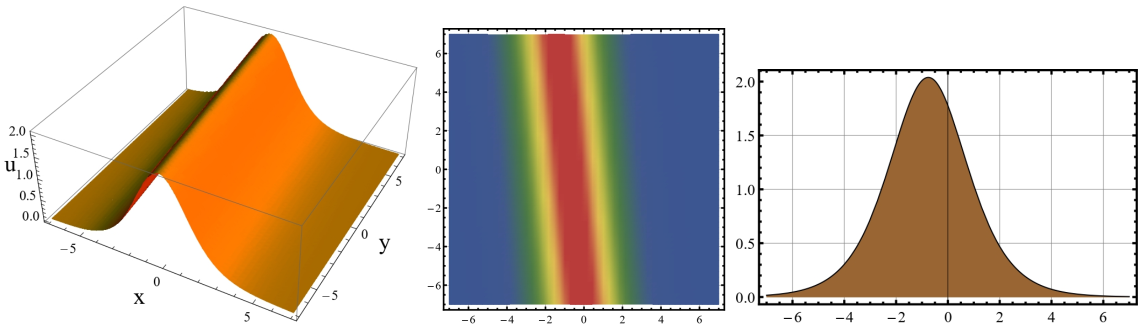

Figure 1: Travelling wave profile of solution (26) for certain parametric values

The solution (26) is graphically shown in Figure 1 with different parametric values. The values in the first figure (left) and second figure (middle) are taken to be $\kappa = 1.1$, $n = 2$, $\omega = 0.99$, $\rho = 0.5$, $b = 1$, $c = 0.85$, $d = 0.5$ where $t = 1$, $z = 0.85$ and $-10 \leq x, y \leq 10$. The values in the third figure (right) are $\kappa = 1.1$, $n = 2$, $\omega = 0.99$, $\rho = 0.5$, $b = 1$, $c = 0.85$, $d = 0.5$ where $t = 1$, $z = 0.85$, $y = 0$ and $-10 \leq x \leq 10$.

Soliton solution of equation (6) for $n = 1$

By assuming

$$u = Q(p), \;\; p = a_1 t + a_2 x + a_3 y + a_4 z, \tag{27}$$

the gnKP-BBM equation (6) with $n = 1$, transforms into a fourth-order NLODE

$$AQ'' - B(Q'^2 + QQ'') + CQ'''' = 0, \tag{28}$$

where $A = a_1a_2 + aa_2^2 + da_3 + ea_4^2$, $B = -ba_2^2$ and $C = ca_1a_2^3$. Integration of (28) yields

$$AQ' - BQQ' + CQ''' + K_1 = 0 \tag{29}$$

with K_1 being a constant. To further integrate this equation, we need to take $K_1 = 0$. Thus, we obtain

$$AQ - \frac{1}{2}BQ^2 + CQ'' + K_2 = 0, \tag{30}$$

where K_2 is an integration constant. Integrating (30) we get

$$\frac{1}{2}AQ^2 - \frac{1}{6}BQ^3 + \frac{1}{2}CQ'^2 + K_2Q + K_3 = 0 \tag{31}$$

with K_3 being a constant. To obtain soliton solutions from (31) we ought to take $K_2 = K_3 = 0$. Integrating the resultant equation, we obtain

$$Q'^2 = EQ^3 + FQ^2, \tag{32}$$

which on integrating gives

$$Q(p) = \frac{F}{E}\operatorname{sech}^2\left\{\frac{\sqrt{F}}{2}(p+K_4)\right\}, \tag{33}$$

where $E = B/(3C), F = -A/C$ and K_4 a constant. This implies that

$$u(t,x,y,z) = -\frac{3ca_1a_2F}{b}\operatorname{sech}^2\left\{\frac{\sqrt{F}}{2}(p+K_4)\right\}, \tag{34}$$

where $F = -(a_1a_2 + aa_2^2 + da_3 + ea_4^2)/(ca_1a_2^3)$, $p = a_1t + a_2x + a_3y + a_4z$ and K_4 is a constant.

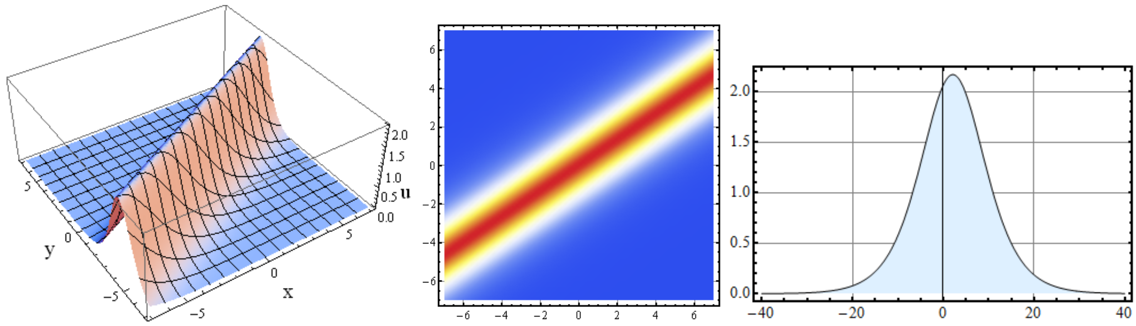

Figure 2: The profile of the travelling wave solution (34)

The travelling wave solution (34) is graphically illustrated in Figure 2 with different parametric values. In the first figure (left) and second figure (middle) the values taken are $a_1 = 40.45$, $a_2 = 50.05$, $a_3 = 26$, $a_4 = 4$, $a = 15$, $b = 20.05$, $c = -80$, $d = -240$, $e = -4$, $\rho = 5$, $K_4 = -15$ where $t = 3$, $z = 3$, and $-7 \leq x, y \leq 7$. The values in the second figure (middle) are $a_1 = 40.45$, $a_2 = 50.05$, $a_3 = 26$, $a_4 = 4$, $a = 15$, $b = 20.05$, $c = -80$, $d = -240$, $e = -4$, $\rho = 5$, $K_4 = -15$ where $t = 3$, $z = 3$, and $-7 \leq x, y \leq 7$. The values in the third figure (right) are $a_1 = 100.45$, $a_2 = 50.05$, $a_3 = 26$, $a_4 = 4$, $a = 15$, $b = 20.05$, $c = -80$, $d = -240$, $e = -4$, $\rho = 5$, $K_4 = -15$ where $t = 22$, $z = 80$, $y = 12$ and $-7 \leq x \leq 7$.

Solution of (31) using direct integration

We now compute exact solution of (31) by applying the direct integration. Equation (31) can be written as

$$Q'^2 = \frac{B}{3C}Q^3 - \frac{A}{C}Q^2 - \frac{2K_2}{C}Q - \frac{2K_3}{C}, \tag{35}$$

where K_2, K_3 are constants. To obtain solution of (35), we assume that α_1, α_2 and α_3 are roots of cubic polynomial equation

$$Q^3 - \frac{3A}{B}Q^2 - \frac{6K_2}{B}Q - \frac{6K_3}{B} = 0, \tag{36}$$

with $\alpha_1 > \alpha_2 > \alpha_3$. Thus, equation (35) can be rewritten as

$$Q'^2 = \frac{B}{3C}(Q-\alpha_1)(Q-\alpha_2)(Q-\alpha_3),$$

whose solution is

$$Q(p) = \alpha_2 + (\alpha_1 - \alpha_2)\operatorname{cn}^2\left\{\sqrt{\frac{B(\alpha_1-\alpha_3)}{4C}}(p-p_0), R^2\right\}, \ R^2 = \frac{\alpha_1-\alpha_2}{\alpha_1-\alpha_3}, \tag{37}$$

with p_0, constant and (cn) is known as Jacobi cosine function. As a result, exact solution of gnKP-BBM equation (6) is

$$u(t,x,y,z) = \alpha_2 + (\alpha_1 - \alpha_2)\mathrm{cn}^2 \left\{ \sqrt{\frac{b(\alpha_3 - \alpha_1)}{4ca_1}}(a_1 t + a_2 x + a_3 y + a_4 z - p_0), \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3} \right\}. \quad (38)$$

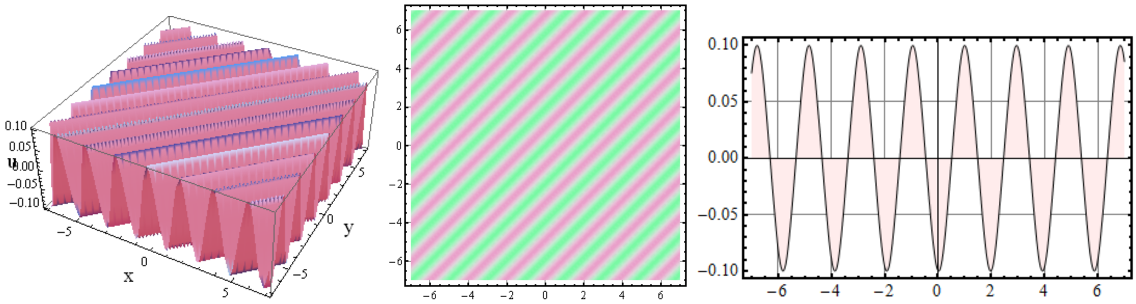

Figure 3: The profile of the periodic solution (38)

The Jacobi cosine function (38) solution is graphically illustrated in Figure 3 with different parametric values. Values in the first figure (left) and second figure (middle) are taken as $\alpha_1 = 100$, $\alpha_2 = 50.05$, $\alpha_3 = -60$, $\rho_0 = 1$, $a_1 = a_2 = a_3 = a_4 = 1$ where $t = -14$, $z = 1$ and $-7 \leq x, y \leq 7$. The values in the third figure (right) are $\alpha_1 = 100$, $\alpha_2 = 50.05$, $\alpha_3 = -60$, $\rho_0 = 1$, $a_1 = a_2 = a_3 = a_4 = 1$ where $t = -14$, $y = 0$, $z = 1$ and $-7 \leq x \leq 7$.

Solutions of (6) for $n = 1$ by (G'/G)-expansion technique

We now utilize (G'/G)-expansion technique Wang et al. (2008); Adeyemo & Khalique (2021) to acquire exact solutions of the gnKP-BBM equation (6). For this reason, we contemplate that a solution of equation (6) is established as

$$F(p) = \sum_{i=0}^{N} A_i \left(\frac{G'(p)}{G(p)}\right)^i, \quad (39)$$

with A_i being parameters to be found and N is a positive nonzero constant. The function $G(p)$ solves

$$G'' + \lambda G' + \mu G = 0 \quad (40)$$

with μ, λ constants. The balancing procedure (Wang et al., 2008) when applied to (28) gives $N = 2$ and so (39) gives

$$F(p) = A_0 + A_1 \left(\frac{G'(p)}{G(p)}\right) + A_2 \left(\frac{G'(p)}{G(p)}\right)^2. \quad (41)$$

Substituting (41) into (28), engaging (40) and subsequently comparing the coefficients of powers of (G'/G), leads to seven algebraic equations in A_0, A_1, A_2, given by

$$
\begin{aligned}
&bA_2 + 12ca_1a_3 = 0,\\
&2bA_1A_2 + 3b\lambda A_2^2 + 4cA_1a_1a_3 + 56c\lambda A_2a_1a_3 = 0,\\
&6dA_2a_2^2 + 6A_2a_1a_3 + 3bA_1^2a_3^2 + 6aA_2a_3^2 + 6bA_0A_2a_3^2 + 21b\lambda A_1A_2a_3^2 + 8b\lambda^2A_2^2a_3^2\\
&\; + 16b\mu A_2^2a_3^2 + 60c\lambda A_1a_1a_3^3 + 330c\lambda^2A_2a_1a_3^3 + 240c\mu A_2a_1a_3^3 + 6eA_2a_4^2 = 0,\\
&d\lambda\mu A_1a_2^2 + 2d\mu^2A_2a_2^2 + \lambda\mu A_1a_1a_3 + 2\mu^2A_2a_1a_3 + a\lambda\mu A_1a_3^2 + b\lambda\mu A_0A_1a_3^2 + b\mu^2A_1^2a_3^2\\
&\; + 2a\mu^2A_2a_3^2 + 2b\mu^2A_0A_2a_3^2 + c\lambda^3\mu A_1a_1a_3^3 + 8c\lambda\mu^2A_1a_1a_3^3 + e\lambda\mu A_1a_4^2 + 16c\mu^3A_2a_1a_3^3\\
&\; + 14c\lambda^2\mu^2A_2a_1a_3^3 + 2e\mu^2A_2a_4^2 = 0,\\
&2dA_1a_2^2 + 10d\lambda A_2a_2^2 + 2A_1a_1a_3 + 10\lambda A_2a_1a_3 + 2aA_1a_3^2 + 2bA_0A_1a_3^2 + 5b\lambda A_1^2a_3^2\\
&\; + 10a\lambda A_2a_3^2 + 10b\lambda A_0A_2a_3^2 + 9b\lambda^2A_1A_2a_3^2 + 18b\mu A_1A_2a_3^2 + 14b\lambda\mu A_2^2a_3^2\\
&\; + 50c\lambda^2A_1a_1a_3^3 + 40c\mu A_1a_1a_3^3 + 130c\lambda^3A_2a_1a_3^3 + 440c\lambda\mu A_2a_1a_3^3 + 2eA_1a_4^2\\
&\; + 10e\lambda A_2a_4^2 = 0,\\
&d\lambda^2A_1a_2^2 + 2d\mu A_1a_2^2 + 6d\lambda\mu A_2a_2^2 + \lambda^2A_1a_1a_3 + 2\mu A_1a_1a_3 + 6\lambda\mu A_2a_1a_3 + a\lambda^2A_1a_3^2\\
&\; + 2a\mu A_1a_3^2 + b\lambda^2A_0A_1a_3^2 + 2b\mu A_0A_1a_3^2 + 3b\lambda\mu A_1^2a_3^2 + 6a\lambda\mu A_2a_3^2 + 6b\lambda\mu A_0A_2a_3^2\\
&\; + 6b\mu^2A_1A_2a_3^2 + c\lambda^4A_1a_1a_3^3 + 22c\lambda^2\mu A_1a_1a_3^3 + 16c\mu^2A_1a_1a_3^3 + 30c\lambda^3\mu A_2a_1a_3^3\\
&\; + 120c\lambda\mu^2A_2a_1a_3^3 + e\lambda^2A_1a_4^2 + 2e\mu A_1a_4^2 + 6e\lambda\mu A_2a_4^2 = 0,\\
&3d\lambda A_1a_2^2 + 4d\lambda^2A_2a_2^2 + 8d\mu A_2a_2^2 + 3\lambda A_1a_1a_3 + 4\lambda^2A_2a_1a_3 + 8\mu A_2a_1a_3 + 3a\lambda A_1a_3^2\\
&\; + 3b\lambda A_0A_1a_3^2 + 2b\lambda^2A_1^2a_3^2 + 4b\mu A_1^2a_3^2 + 4a\lambda^2A_2a_3^2 + 8a\mu A_2a_3^2 + 4b\lambda^2A_0A_2a_3^2\\
&\; + 8b\mu A_0A_2a_3^2 + 15b\lambda\mu A_1A_2a_3^2 + 6b\mu^2A_2^2a_3^2 + 15c\lambda^3A_1a_1a_3^3 + 60c\lambda\mu A_1a_1a_3^3\\
&\; + 16c\lambda^4A_2a_1a_3^3 + 232c\lambda^2\mu A_2a_1a_3^3 + 136c\mu^2A_2a_1a_3^3 + 3e\lambda A_1a_4^2 + 4e\lambda^2A_2a_4^2\\
&\; + 8e\mu A_2a_4^2 = 0.
\end{aligned}
$$

Using Maple, solution to the above equations is

$$
\begin{aligned}
&A_0 = -\frac{da_2^2 + a_1a_3 + aa_3a_3^2 + c\lambda^2a_1a_3^3 + 8c\mu a_1a_3^3 + ea_4^2}{ba_3^2},\\
&A_1 = -\frac{12\lambda ca_1a_3}{b},\; A_2 = -\frac{12ca_1a_3}{b}.
\end{aligned}
$$

Thus, we can write down the solutions of gnKP-BBM equation (6) as given in the following three cases:

1. For $\lambda^2 - 4\mu > 0$, we secure

$$
\begin{aligned}
u(t,x,y,z) = A_0 + A_1\Big\{ -\frac{\lambda}{2} + \delta\Big(\frac{C_1\sinh(\delta p) + C_2\cosh(\delta p)}{C_1\cosh(\delta p) + C_2\sinh(\delta p)}\Big)\Big\}\\
+ A_2\Big\{ -\frac{\lambda}{2} + \delta\Big(\frac{C_1\sinh(\delta p) + C_2\cosh(\delta p)}{C_1\cosh(\delta p) + C_2\sinh(\delta p)}\Big)\Big\}^2
\end{aligned}
\tag{42}
$$

with $p = a_1t + a_2x + a_3y + a_4z$, $\delta = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1, C_2 being constants. These are the hyperbolic function solutions of (6).

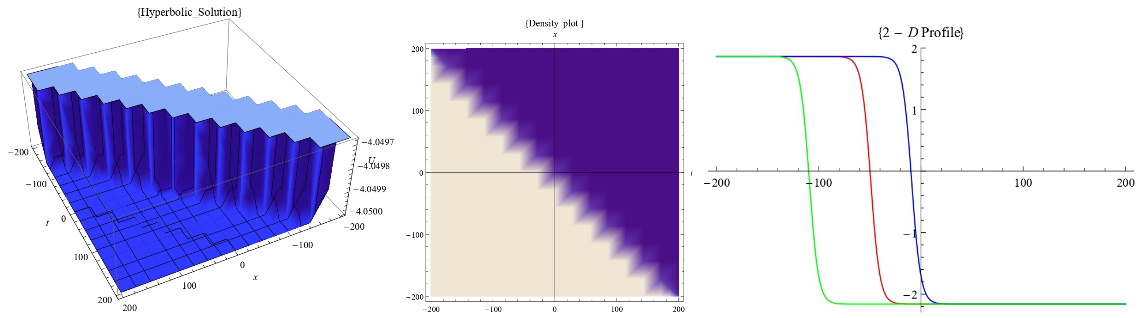

Figure 4: The profile of the hyperbolic function solution (42)

The hyperbolic function solution (42) is portrayed graphically in Figure 4 with distict values. The values in the first figure (left) and second figure (middle) are taken to be $C_1 = 1, C_2 = 1.5, \lambda = 0.5, \mu = 0.05, a_1 = a_2 = a_3 = a_4 = 1, a = b = c = d = e = 1$, where $z = 1$, $y = 1$ and $-10 \leq t, x \leq 10$. The values in the third figure (right) are $C_1 = 1, C_2 = 1.5, \lambda = 0.5, \mu = 0.05, a_1 = a_2 = a_3 = a_4 = 1, a = b = c = d = e = 1$, where $z = 1$, $y = 1$ and $-200 \leq x \leq 200$ for $t = 0, t = 40, t = 100$.

2. For $\lambda^2 - 4\mu < 0$, we get

$$u(t,x,y,z) = A_0 + A_1\left\{ -\frac{\lambda}{2} + \delta\left(\frac{-C_1\sin(\delta p) + C_2\cos(\delta p)}{C_1\cos(\delta p) + C_2\sin(\delta p)}\right)\right\} + A_2\left\{ -\frac{\lambda}{2} + \delta\left(\frac{-C_1\sin(\delta p) + C_2\cos(\delta p)}{C_1\cos(\delta p) + C_2\sin(\delta p)}\right)\right\}^2, \tag{43}$$

with $p = a_1t + a_2x + a_3y + a_4z$, $\delta = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1, C_2 constants. These are the trigonometric function solutions of (6).

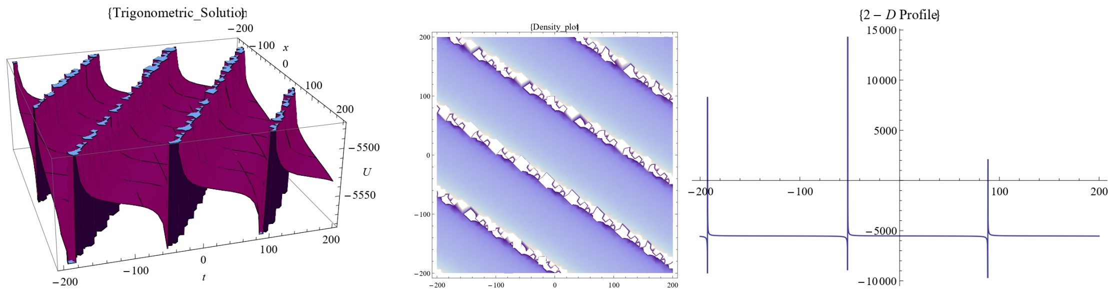

Figure 5: The profile of the trigonometric function solution (43)

The trigonometric function solution (43) is shown graphically in Figure 5 with different values. The values in the first figure (left) and second figure (middle) are $C_1 = 1, C_2 = 1.5, \lambda = 0.5, \mu = 0.05, a_1 = -8, a_2 = 0.20, a_3 = 0.15, a_4 = 5, a = 0.25, b = 0.10, c = 0.14, d = 0.55, e = 0.45$ where $z = 1$, $y = 1$, and $-200 \leq t, x \leq 200$. The values in the third figure (right) are $C_1 = 1, C_2 = 1.5, \lambda = 0.5, \mu = 0.05, a_1 = -8, a_2 = 0.20, a_3 = 0.15, a_4 = 5, a = 0.25, b = 0.10, c = 0.14, d = 0.55, e = 0.45$ where $z = 1$, $y = 1$, $t = 0$ and $-200 \leq x \leq 200$.

3. For $\lambda^2 - 4\mu = 0$, we obtain

$$u(t,x,y,z) = A_0 + A_1\left\{-\frac{\lambda}{2} + \frac{C_2}{C_1 + pC_2}\right\} + A_2\left\{-\frac{\lambda}{2} + \frac{C_2}{C_1 + pC_2}\right\}^2 \tag{44}$$

with $p = a_1t + a_2x + a_3y + a_4z$, C_1, C_2 constants. This is the rational function solution of (6).

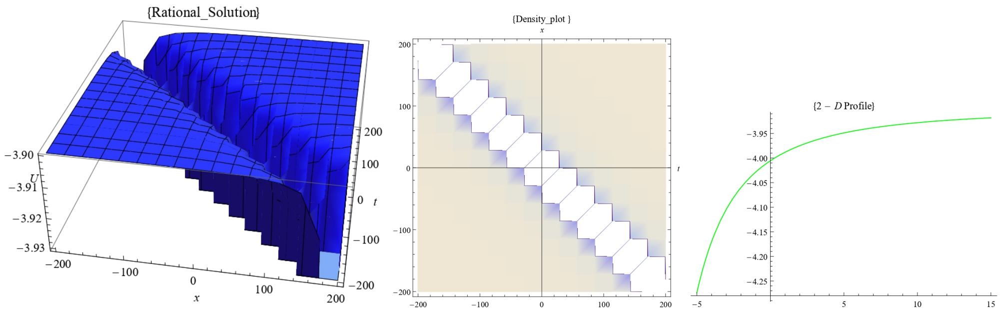

Figure 6: The profile of the rational function solution (44)

The rational function solution (44) is depicted graphically in Figure 6 with unalike values. The values in the first figure (left) and second figure (middle) are $C_1 = 1, C_2 = 1.5, \lambda = 0.5, \mu = 0.05, a_1 = 1, a_2 = 2, a_3 = 1, a_4 = 1, a = b = c = d = e = 1$ where $z = 1$, $y = 1$ and $-200 \leq t, x \leq 200$. The values in the last figure (right) are $C_1 = 1, C_2 = 1.5, \lambda = 0.5, \mu = 0.05, a_1 = 1, a_1 = 2, a_3 = 1, a_4 = 1, a = b = c = d = e = 1$ where $t = 10, y = 1, z = 1$ and $-5 \leq x \leq 5$.

2.1.2 Case 2 : Symmetry reductions of (6) using H_5

We now make use of the symmetry H_5. The related Lagrange system of symmetry H_5 gives the following invariants:

$$f = t,\ g = x,\ h = ey^2 + dz^2,\ W = u. \tag{45}$$

Using these invariants the gnKP-BBM equation (6) transforms to

$$W_{fg} + aW_{gg} + nbW^{n-1}W_g^2 + bW^nW_{gg} + cW_{gg} + 4edW_{hh}h + 4edW_h = 0. \tag{46}$$

Equation (46) has symmetries that include the two translation symmetries, namely,

$$G_1 = \frac{\partial}{\partial f},\ G_2 = \frac{\partial}{\partial g}.$$

The symmetry $G = G_1 + \epsilon G_2$, where ϵ is a constant, gives two invariants

$$\tau = f - \epsilon g, \quad \psi = W \tag{47}$$

and these invariants, transfroms equation (46) into a second-order NLODE, given as

$$(c\epsilon + a\epsilon - 1)\,\epsilon\psi\psi'' + nb\epsilon^2\psi^n\psi'^2 + b\epsilon^2\psi^{n+1}\psi'' = 0. \tag{48}$$

Thus, we have successfully performed symmetry reductions on the gnKP-BBM equation (6) and reduced it to a second-order NODE.

3 Conservation laws of (6)

Using the Ibragimov's method Adeyemo & Khalique (2007), we construct conservation laws of (6). This theorem can be applied to any system of differential equations and does not require an existence of a Lagrangian. To begin, we write down adjoint equation of (6) as

$$F^* \equiv \frac{\delta}{\delta u}\{v(u_{tx} + au_{xx} + b(u^nu_x)_x + cu_{txxx} + du_{yy} + eu_{zz}\} = 0. \tag{49}$$

In this case

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}+D_y^2\frac{\partial}{\partial u_{yy}}+D_z^2\frac{\partial}{\partial u_{zz}}+D_tD_x\frac{\partial}{\partial u_{tx}}+D_tD_x^3\frac{\partial}{\partial u_{txxx}} \tag{50}$$

is the Euler operator and D_t, D_x, D_y and D_z are given as

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t}+v_t\frac{\partial}{\partial v}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+v_{tt}\frac{\partial}{\partial v_t}+v_{tx}\frac{\partial}{\partial v_x}+\cdots,\\
D_x &= \frac{\partial}{\partial x}+v_x\frac{\partial}{\partial v}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+v_{xx}\frac{\partial}{\partial v_x}+v_{xt}\frac{\partial}{\partial v_t}\cdots,\\
D_y &= \frac{\partial}{\partial y}+v_y\frac{\partial}{\partial v}+u_y\frac{\partial}{\partial u}+u_{yy}\frac{\partial}{\partial u_y}+u_{yt}\frac{\partial}{\partial u_t}+v_{yy}\frac{\partial}{\partial v_y}+v_{yt}\frac{\partial}{\partial v_t}\cdots,\\
D_z &= \frac{\partial}{\partial z}+u_z\frac{\partial}{\partial u}+v_z\frac{\partial}{\partial v}+u_{zz}\frac{\partial}{\partial u_z}+v_{zz}\frac{\partial}{\partial v_z}+u_{zt}\frac{\partial}{\partial u_t}+v_{zt}\frac{\partial}{\partial v_t}\cdots.
\end{aligned}$$

The adjoint equation (49) can be written as

$$F^*\equiv v_{tx}+av_{xx}+bu^nv_{xx}+dv_{yy}+ev_{zz}+cv_{txxx}=0. \tag{51}$$

We now consider equation (6) and its adjoint (51). The Lagrangian of equation (6) is given as

$$\mathcal{L}=v(u_{tx}+au_{xx}+b(u^nu_x)_x+du_{yy}+eu_{zz})+cv_{tx}u_{xx}. \tag{52}$$

Recall that five Lie point symmetries (11) are admitted by (6). Therefore, using the formula in Adeyemo & Khalique (2007), we compute the related conserved vectors for the given Lagrangian. For the corresponding symmetries, we derive the conserved vectors of (6). These are

$$\begin{aligned}
T_1^t &= evu_{zz}+dvu_{yy}+bnu^{n-1}vu_x{}^2+avu_{xx}+bu^nvu_{xx}+\frac{1}{2}v_xu_t+\frac{1}{4}cv_{xxx}u_t\\
&\quad+\frac{1}{2}vu_{tx}-\frac{1}{4}cv_{xx}u_{tx}+\frac{1}{4}cv_xu_{txx}+\frac{3}{4}cvu_{txxx},\\
T_1^x &= av_xu_t-bnu^{n-1}vu_xu_t+bu^nv_xu_t+\frac{1}{2}u_tv_t-avu_{tx}-bu^nvu_{tx}-\frac{1}{2}cu_{tx}v_{tx}\\
&\quad+\frac{1}{4}cv_tu_{txx}+\frac{3}{4}cu_tv_{txx}-\frac{1}{2}vu_{tt}-\frac{1}{4}cv_{xx}u_{tt}+\frac{1}{2}cv_xu_{ttx}-\frac{3}{4}cvu_{ttxx},\\
T_1^y &= dv_yu_t-dvu_{ty},\\
T_1^z &= ev_zu_t-evu_{tz};
\end{aligned}$$

$$\begin{aligned}
T_2^t &= \frac{1}{2}u_xv_x-\frac{1}{2}vu_{xx}-\frac{1}{4}cu_{xx}v_{xx}+\frac{1}{4}cv_xu_{xxx}+\frac{1}{4}cu_xv_{xxx}-\frac{1}{4}cvu_{xxxx},\\
T_2^x &= evu_{zz}+dvu_{yy}+au_xv_x+bu^nu_xv_x+\frac{1}{2}u_xv_t+\frac{1}{4}cu_{xxx}v_t+\frac{1}{2}vu_{tx}\\
&\quad-\frac{1}{4}cv_{xx}u_{tx}-\frac{1}{2}cu_{xx}v_{tx}+\frac{1}{2}cv_xu_{txx}+\frac{3}{4}cu_xv_{txx}+\frac{1}{4}cvu_{txxx},\\
T_2^y &= dv_yu_x-dvu_{xy},\\
T_2^z &= ev_zu_x-evu_{xz};
\end{aligned}$$

$$\begin{aligned}
T_3^t &= \frac{1}{2}u_yv_x-\frac{1}{2}vu_{xy}-\frac{1}{4}cu_{xy}v_{xx}+\frac{1}{4}cv_{,x}u_{xxy}+\frac{1}{4}cu_yv_{xxx}-\frac{1}{4}cvu_{xxxy},\\
T_3^x &= au_yv_x-bnu^{n-1}vu_yu_x+bu^nu_yv_x-avu_{xy}-bu^nvu_{xy}+\frac{1}{2}u_yv_t+\frac{1}{4}cu_{xxy}v_t\\
&\quad-\frac{1}{2}vu_{ty}-\frac{1}{4}cv_{xx}u_{ty}-\frac{1}{2}cu_{xy}v_{tx}+\frac{1}{2}cv_xu_{txy}+\frac{3}{4}cu_yv_{txx}-\frac{3}{4}cvu_{txxy},\\
T_3^y &= evu_{zz}+du_yv_y+bnu^{n-1}vu_x{}^2+avu_{xx}+bu^nvu_{xx}+vu_{tx}+cvu_{txxx},\\
T_3^z &= ev_zu_y-evu_{yz};
\end{aligned}$$

$$\begin{aligned}
T_4^t =& \frac{1}{2}u_z v_x - \frac{1}{2}v u_{xz} - \frac{1}{4}c u_{xz} v_{xx} + \frac{1}{4}c v_x u_{xxz} + \frac{1}{4}c u_z v_{xxx} - \frac{1}{4}c v u_{xxxz},\\
T_4^x =& a u_z v_x - bnu^{n-1} v u_z u_x + bu^n u_z v_x - a v u_{xz} - bu^n v u_{xz} + \frac{1}{2}u_z v_t + \frac{1}{4}c u_{xxz} v_t\\
& - \frac{1}{2}v u_{tz} - \frac{1}{4}c v_{xx} u_{tz} - \frac{1}{2}c u_{xz} v_{tx} + \frac{1}{2}c v_x u_{txz} + \frac{3}{4}c u_z v_{txx} - \frac{3}{4}c v u_{txxz},\\
T_4^y =& d u_z v_y - d v u_{yz},\\
T_4^z =& e u_z v_z + d v u_{yy} + bnu^{n-1} v {u_x}^2 + a v u_{xx} + bu^n v u_{xx} + v u_{tx} + c v u_{txxx};
\end{aligned}$$

$$\begin{aligned}
T_5^t =& \frac{1}{2}dz u_y v_x - \frac{1}{2}ey u_z v_x + \frac{1}{2}ey v u_{xz} - \frac{1}{2}dz v u_{xy} + \frac{1}{4}cey u_{xz} v_{xx} - \frac{1}{4}cdz u_{xy} v_{xx}\\
& - \frac{1}{4}cey v_x u_{xxz} + \frac{1}{4}cdz v_x u_{xxy} - \frac{1}{4}cey u_z v_{xxx} + \frac{1}{4}cdz u_y v_{xxx} + \frac{1}{4}cey v u_{xxxz}\\
& - \frac{1}{4}cdz v u_{xxxy},\\
T_5^x =& nbeyu^{n-1} v u_z u_x - bdnzu^{n-1} v u_y u_x - aey u_z v_x - beyu^n u_z v_x + adz u_y v_x\\
& + bdzu^n u_y v_x + aey v u_{xz} + beyu^n v u_{xz} - adz v u_{xy} - bdzu^n v u_{xy} - \frac{1}{2}ey u_z v_t\\
& + \frac{1}{2}dz u_y v_t - \frac{1}{4}cey u_{xxz} v_t + \frac{1}{4}cdz u_{xxy} v_t + \frac{1}{2}ey v u_{tz} + \frac{1}{4}cey v_{xx} u_{tz} - \frac{1}{2}dz v u_{ty}\\
& - \frac{1}{4}cdz v_{xx} u_{ty} + \frac{1}{2}cey u_{xz} v_{tx} - \frac{1}{2}cdz u_{xy} v_{tx} - \frac{1}{2}cey v_x u_{txz} + \frac{1}{2}cdz v_x u_{txy}\\
& - \frac{3}{4}cey u_z v_{txx} + \frac{3}{4}cdz u_y v_{txx} + \frac{3}{4}cey v u_{txxz} - \frac{3}{4}cdz v u_{txxy},\\
T_5^y =& dev u_z + dez v u_{zz} - dey u_z v_y + d^2 z u_y v_y + dey v u_{yz} + bdnzu^{n-1} v {u_x}^2 + adz v u_{xx}\\
& + bdzu^n v u_{xx} + dz v u_{tx} + cdz v u_{txxx},\\
T_5^z =& dez v_z u_y - e^2 y u_z v_z - dev u_y - dez v u_{yz} - dey v u_{yy} - benyu^{n-1} v {u_x}^2 - aey v u_{xx}\\
& - beyu^n v u_{xx} - ey v u_{tx} - cey v u_{txxx}.
\end{aligned}$$

4 Results and discussion

In this research, firstly we introduced the gnKP-BBM equation with power law nonlinearity (6) and thereafter investigated it using Lie group analysis. Utilizing its translation symmetries we performed symmetry reductions which led to the fourth-order NLODE (20). Kudryashov's method was then applied to (20), which gave us an exact solution to (6), that is presented for the first time. Moreover, conservation laws were also derived for the first time with the help of Ibragimov's theorem.

Secondly, this study investigated the exact solutions of (6) for $n = 1$. To achieve this, symmetry reductions, direct integration, the Jacobi cosine approach and the (G'/G) expansion method were used. According to our research, all solutions obtained of (6) for $n = 1$ have never been presented before in the literature and are given here for the first time. To the best of our knowledge, no one has addressed exact solutions for equation (6) when $n = 1$ using these methods.

Furthermore, the dynamics of the exact solutions obtained in this work were depicted using suitable graphs which were also discussed in detail. See Figures 1-6.

5 Conclusions

Through the use of various techniques, few researchers have identified a few different types of solutions to the equation (6) which is an equation for small amplitude long waves in shallow water that mainly move in the x direction, but in this most recent research, we have discovered more general and new solutions for equation (6). We obtain exact solutions for equation (6) by the use of Lie symmetry reductions, direct integration, Kudryashov's method, Jacobi cosine approach and the (G'/G)-expansion method. Moreover, we derived conservation laws of (6) by using Ibragimov's theorem. These conversation laws are linked respectively to the conservation of energy and momentum which holds broad significance across scientific and engineering disciplines.

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